E Covering Spaces

a **covering space** of a space X is a pair $(\tilde{X}, p)$ where

$\tilde{X}$ is a space and

$p: \tilde{X} \to X$ is a continuous map such that every point $x \in X$ has an evenly covered neighborhood

an open set U is called **evenly covered** if

$p^{-1}(U)$ = disjoint union of open sets $\{U_\alpha\}$ in $\tilde{X}$

such that $p|_{U_\alpha}: U_\alpha \to U$ is a homeomorphism $\forall \alpha$

examples:

1) a homeomorphism $p: \tilde{X} \to X$ is a covering space

2) $p: \mathbb{R} \to S^1 : t \mapsto (\cos 2\pi t, \sin 2\pi t)$ is a covering space

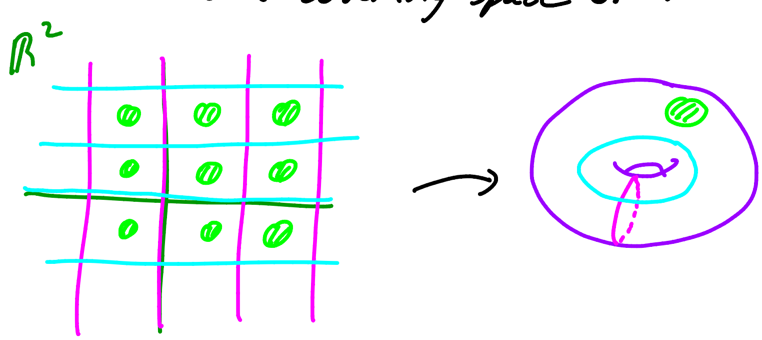
since we showed $A = S^1 - \{(1,0)\}$ and $B = S^1 - \{(-1,0)\}$ are evenly covered

exercise: if $(\tilde{X}, p)$ is a covering space of X and $(\tilde{Y}, q)$ " " " " Y

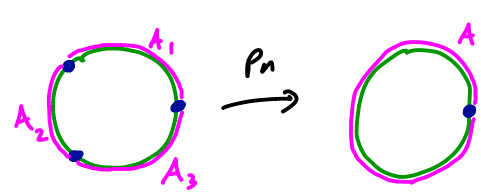
then $(\tilde{X} \times \tilde{Y}, p \times q)$ is a covering space of $X \times Y$

so $p: \mathbb{R}^2 \to T^2 : (x,y) \to ((\cos 2\pi x, \sin 2\pi x), (\cos 2\pi y, \sin 2\pi y))$

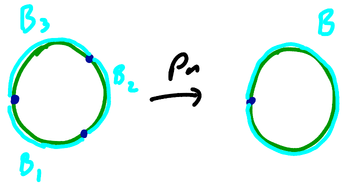
is a covering space of T^2

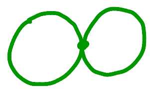


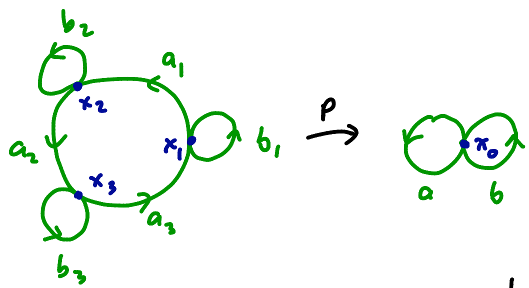
3) $p_n: S^1 \to S^1 : \theta \mapsto n\theta$



wrap S^1 n times around S^1

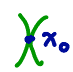


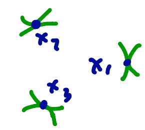
4) $X =$  wedge of 2 circles



<u>note:</u> a evenly covered

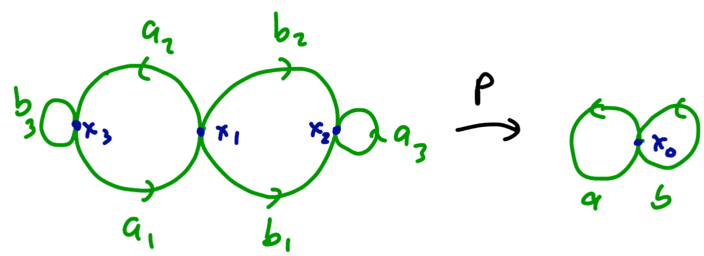
b evenly covered

x_0 has nbhd 

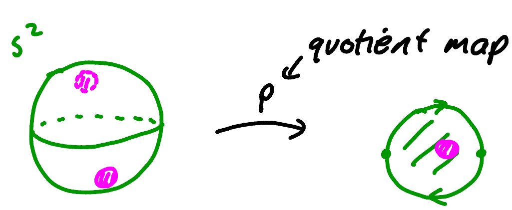
evenly covered by 

<u>exercise:</u> explicitly write out p

similarly another cover of X is



5) $\mathbb{R}P^2 = S^2/\sim$ points in S^2 $\sim$ to antipode



similarly $S^n \to \mathbb{R}P^n$ a covering map

lemma 19:

> let $(\tilde{X}, p)$ be a covering space of a connected space X
> the cardinality $|p^{-1}(x)|$ is independent of X

$|p^{-1}(x)|$ is called the <u>degree</u> of the covering space

Proof: for some $x_0 \in X$, let $k = |p^{-1}(x_0)|$

let $A = \{x \in X : |p^{-1}(x)| = k\}$

if $x \in A$ then let U be a evenly covered nbhd of x

so $p^{-1}(U) = \{U_1, \dots U_k\}$

$\therefore |p^{-1}(x')| = k \quad \forall x' \in U$

$\therefore U \subset A$ and A open

similarly $X - A$ open so A closed

$\therefore X = A$ since X connected $\blacksquare$

If $(\tilde{X}, p)$ a covering space of X

$f : Y \to X$ a continuous map

then $\tilde{f} : Y \to \tilde{X}$ a <u>lift</u> of f if $p \circ \tilde{f} = f$

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p"] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

if $f(y_0) = x_0$ and $\tilde{x}_0 \in \tilde{X}$ s.t. $p(\tilde{x}_0) = x_0$, then $\tilde{f}$ is a <u>lift of f based at $\tilde{x}_0$</u>
if $\tilde{f}(y_0) = \tilde{x}_0$

lemma 20:

> $(\tilde{X}, p)$ a covering space of X, $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$
>
> path lifting → a) each path $\gamma : [0,1] \to X$ based at x_0 has a unique lift to a path $\tilde{\gamma} : [0,1] \to X$ based at $\tilde{x}_0$
>
> homotopy lifting → b) if $H : Y \times [0,1] \to X$ is a homotopy with $h_0(y) = H(y,0)$ and $\tilde{h}_0 : Y \to \tilde{X}$ a lift of h_0 then $\exists$ a unique homotopy $\tilde{H} : Y \times [0,1] \to \tilde{X}$ s.t. $\tilde{h}_0(y) = \tilde{H}(y,0)$

Proof: same as proof of lemma 12 (exercise) $\blacksquare$

lemma 21:

If $(\tilde{X}, p)$ is a path connected covering space of X
and $x_0 \in X$, $\tilde{x}_0 \in p^{-1}(x_0)$

then $p_*: \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0)$

(1) is injective

(2) its image is the set of loops in $\pi_1(X, x_0)$ that when lifted to paths in $\tilde{X}$ based at x_0, they are loops

(3) $[\pi_1(X, x_0) : \pi_1(\tilde{X}, \tilde{x}_0)] =$ degree of $(\tilde{X}, p)$

examples:

1) $p: \mathbb{R} \to S^1$

$$\begin{array}{ccc} p_*: \pi_1(\mathbb{R}) & \to & \pi_1(S^1) \\ \shortparallel & & \cong \\ \{e\} & & \mathbb{Z} \end{array}$$

no non-trivial loop in S^1 lifts to a loop in $\mathbb{R}$

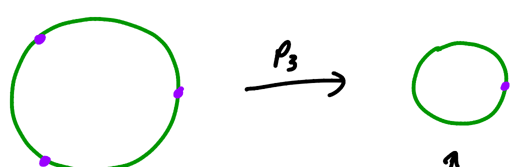
degree $= \infty = [\mathbb{Z} : \{0\}]$

2) $p_n: S^1 \to S^1$

$$\begin{array}{ccc} (p_n)_*: \pi_1(S^1) & \to & \pi_1(S^1) \\ \cong & & \cong \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} \\ 1 & \longmapsto & n \end{array}$$

so $\mathrm{im}(p_n)_* = n\mathbb{Z}$

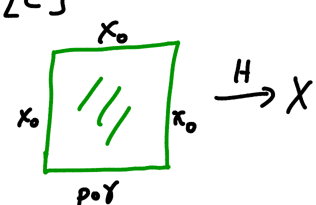
degree $p_n = n = [\mathbb{Z} : n\mathbb{Z}]$



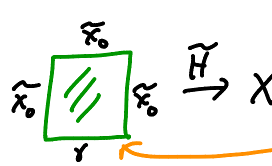
any loop in that wraps a multiple of 3 times lifts to a loop

Proof: 1) Suppose $p_*([\gamma]) = [e]$

so $\exists$ a homotopy



lemma 20, b) $\Rightarrow \exists$ homotopy



why is this γ?

<u>note</u>: $\tilde{H}(0,t)$ lift of $H(0,t) = x_0$ so constantly $\tilde{x}_0$

similarly $\tilde H(1,t)=\tilde x_0=\tilde H(s,1)$

so $\gamma\sim e_{\tilde x_0}$ by $\tilde H$ and $[\gamma]=[e]$

(2) clearly if $[\gamma]\in\pi_1(X,x_0)$ and γ lifts to a loop $\tilde\gamma$ based at $\tilde x_0$

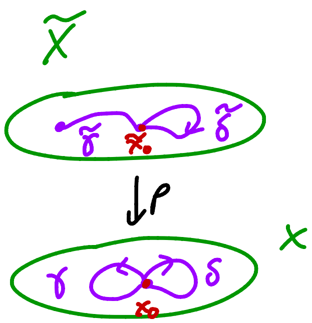
then $p_*([\tilde\gamma])=\gamma$

and if $[\gamma]=p_*([\eta])$, then $\gamma\sim p\circ\eta$ $\therefore$ by lemma 20, b)

the lift $\tilde\gamma$ of γ based at $\tilde x_0$ is homotopic to η (rel end pts)

so $\tilde\gamma$ a loop

(3) let $H=p_*(\pi_1(\tilde X,x_0))$



if $[\gamma]\in\pi_1(X,x_0)$ and $[\delta]\in H$, then note

if $\tilde\gamma$ a lift of γ based at $\tilde x_0$

$\widetilde{\delta*\gamma}$ " " $\delta*\gamma$ " " $\tilde x_0$

then $\tilde\gamma(1)=\widetilde{\delta*\gamma}(1)$ since $\tilde\delta$ is a loop (and $\widetilde{\delta*\gamma}=\tilde\delta*\tilde\gamma$)

$\therefore$ we get a map $\phi:\{\text{right cosets of } H\}\longrightarrow p^{-1}(x_0)$

$$H[\gamma]\longmapsto\tilde\gamma(1)$$

that is well-defined

if $\tilde x_1\in p^{-1}(x_0)$ then let $\tilde\gamma$ be a path $\tilde x_0$ to $\tilde x_1$

$\gamma=p\circ\tilde\gamma$ is a loop in X based at x_0

and $\phi(H[\gamma])=\tilde\gamma(1)=\tilde x_1$

so ϕ onto

now suppose $\phi(H[\gamma])=\phi(H[\eta])$

so if $\tilde\gamma,\tilde\eta$ are lifts of γ,η based at $\tilde x_0$

then $\tilde\gamma(1)=\tilde\eta(1)$

$\therefore$ $\tilde\gamma*\bar{\tilde\eta}$ is a loop in $\tilde X$ and so $p_*([\tilde\gamma*\bar{\tilde\eta}])\in H$

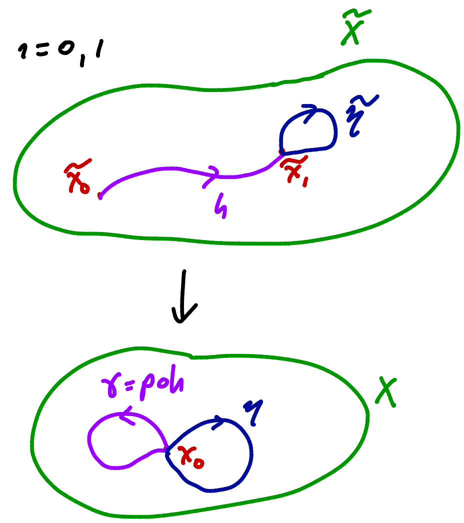
but $p_*([\tilde\gamma*\bar{\tilde\eta}])=[\gamma]*[\eta]^{-1}\Rightarrow H[\gamma]=H[\eta]$

i.e. ϕ is one-to-one ∎

lemma 22:

> If $(\tilde{X},p)$ is a path connected covering space of X and $x_0 \in X$ then
> $$\{p_*(\pi_1(\tilde{X},\tilde{x}))\}_{\tilde{x}\in p^{-1}(x_0)}$$
> is a conjugacy class of subgroups of $\pi_1(X,x_0)$

Proof: let $\tilde{x}_0, \tilde{x}_1 \in p^{-1}(x_0)$ and set $H_i = p_*(\pi_1(\tilde{X},x_i))$ $i=0,1$



let $h:[0,1]\to\tilde{X}$ be a path $\tilde{x}_0$ to $\tilde{x}_1$

then $\gamma = p\circ h$ is a loop in X

if $[\eta]\in H_1$ then η lifts to a loop $\tilde{\eta}$ based at x_1 (by lemma 21)

so $h * \tilde{\eta} * \bar{h}$ is a loop based at $\tilde{x}_0$

$$\therefore\ [\gamma]\cdot[\eta]\cdot[\gamma]^{-1} = [(p\circ h)*(p\circ\tilde{\eta})*\overline{(p\circ h)}]$$
$$= p_*[h*\tilde{\eta}*\bar{h}] \in H_0$$

$\therefore\ [\gamma]H_1[\gamma]^{-1} \subseteq H_0$

similarly $[\gamma]^{-1}H_0[\gamma]\subseteq H_1$ $\therefore$ $H_0 = [\gamma]H_1[\gamma]^{-1}$

now if H is conjugate to H_0, then $\exists[\alpha]\in\pi_1(X,x_0)$ s.t.

$$[\alpha]H[\alpha]^{-1} = H_0$$

if $[\alpha]\in H_0$ then $H = H_0$ and done

if $[\alpha]\notin H_0$ then α lifts to a path $\tilde{\alpha}$ starting at $\tilde{x}_0$

let $\tilde{x}_1 = \tilde{\alpha}(1)$

set $H_1 = p_*(\pi_1(\tilde{X},\tilde{x}_1))$

from above $H_0 = [\alpha]H_1[\alpha]^{-1}$ $\therefore$ $H = H_1$ ∎

Thm 23:

> let $(\tilde{X},p)$ be a covering space of X, $\tilde{x}_0\in\tilde{X}$ and $x_0 = p(\tilde{x}_0)$
> suppose $f:Y\to X$ is a continuous map with
> Y path connected and locally path connected

$y_0 \in Y$ s.t. $f(y_0) = x_0$

Then $\exists$ a lift $\tilde{f}: Y \to \tilde{X}$ $(p \circ \tilde{f} = f)$ s.t. $\tilde{f}(y_0) = \tilde{x}_0$

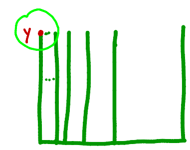
$$\Longleftrightarrow$$

$$f_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$$

(and if lift exists it is unique, see lemma 24 below)

Y is locally path connected if $\forall y \in Y$ and open sets U containing y, $\exists$ an open set V s.t. $y \in V \subset U$ and V is path connected

example: the comb space $C = (\{\frac{1}{n}\} \times [0,1]) \cup (\{0\} \times [0,1]) \cup ([0,1] \times \{0\})$ is path connected but not locally path connected



Proof: $(\Rightarrow)$ $f_*(\pi_1(Y, y_0)) = p_*(\tilde{f}_*(\pi_1(Y, y_0))) \subseteq p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ ✓

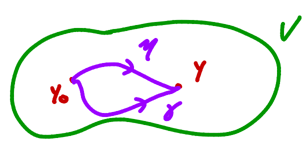
$(\Leftarrow)$ for $y \in Y$ let γ be a path y_0 to y

so $f \circ \gamma$ is a path in X based at x_0

$\exists!$ lift $\widetilde{f \circ \gamma}: [0,1] \to \tilde{X}$ based at $\tilde{x}_0$

define: $\tilde{f}(y) = \widetilde{f \circ \gamma}(1)$ note: if well-defined, then clearly $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = \tilde{x}_0$

Claim: $\tilde{f}$ well-defined



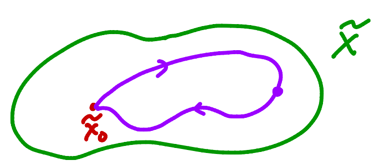
let γ, η be two path y_0 to y

$\gamma * \bar{\eta}$ is a loop in Y based at y_0

so $f_*([\gamma * \bar{\eta}]) \in p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

i.e. $(f \circ \gamma) * (f \circ \bar{\eta})$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$

but $\widetilde{(f \circ \gamma) * (f \circ \bar{\eta})} = \underbrace{\widetilde{f \circ \gamma}}_{\text{lift based at } \tilde{x}_0} * \underbrace{\widetilde{f \circ \bar{\eta}}}_{\text{lift based at } \widetilde{f \circ \gamma}(1)}$



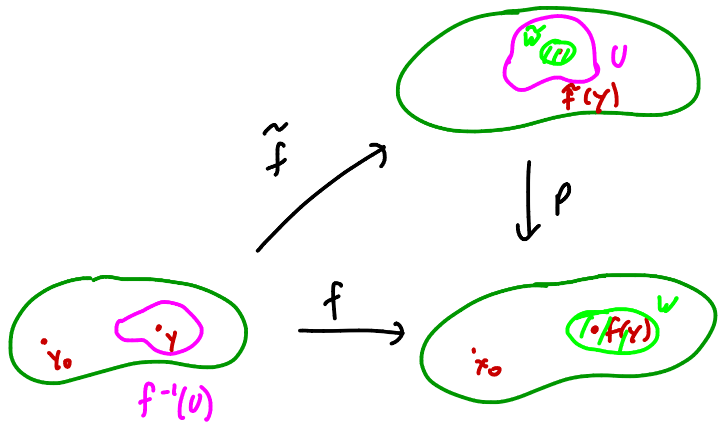
note: 1) $\widetilde{f \circ \bar{\eta}}(1) = \tilde{x}_0$

2) $\widetilde{f \circ \eta} = \overline{\widetilde{f \circ \bar{\eta}}}$ so $\widetilde{f \circ \eta}(1) = \widetilde{f \circ \gamma}(1)$ and $\tilde{f}$ is well-defined!

Claim: $\tilde{f}$ is continuous

given an open set $U \subset \tilde{X}$ we show $\forall y \in \tilde{f}^{-1}(U)$, $\exists$ an open set V s.t. $y \in V \subset \tilde{f}^{-1}(U)$

($\therefore \tilde{f}^{-1}(U)$ open)



idea is on small nbhd of y, $\tilde{f}$ is $f \circ (p|)^{-1}$

let W be an evenly covered nbhd of $f(y)$

and $\tilde{W}$ open set in $\tilde{X}$ s.t. $\tilde{f}(y) \subset \tilde{W} \subset U$

and $p|_{\tilde{W}} : \tilde{W} \to W$ homeomorphism (might need to shrink W)

Y locally path connected $\Rightarrow \exists V$ open in Y s.t. $y \in V \subset f^{-1}(W)$

and V path connected

now fix a path γ from y_0 to y

for any $y' \in V$ let η be a path y to y' in V

so $\gamma * \eta$ is a path y_0 to y'

$\therefore \tilde{f}(y') = \widetilde{f \circ (\gamma * \eta)}(1)$

but if $\widetilde{f \circ \eta}$ is lift of $f \circ \eta$ based at $\widetilde{f \circ \gamma}(1) = \tilde{f}(y)$

then $\widetilde{f \circ (\gamma * \eta)}(1) = \widetilde{f \circ \eta}(1)$

and we know $\widetilde{f \circ \eta} = \underbrace{(p|_{\tilde{W}})^{-1} \circ f \circ \eta}$

since this is a lift and lift is unique!

$\therefore \tilde{f}(y') \in \tilde{W} \subset U$

i.e. $V \subset \tilde{f}^{-1}(U)$

lemma 24:

$(\tilde{X}, p)$ a covering space of X

let $\tilde{f}_1, \tilde{f}_2 : Y \to \tilde{X}$ be two lifts of $f : Y \to X$

if Y is connected and $\tilde{f}_1$ and $\tilde{f}_2$ agree at one point then $\tilde{f}_1 = \tilde{f}_2$

Proof:

let $A = \{ y \in Y \text{ s.t. } \tilde{f}_1(y) = \tilde{f}_2(y) \}$

$A \neq \emptyset$ by assuption

if $y \in A$ then let U be an evenly covered nbhd of $f(y)$

let $\tilde{U}$ be an open set in $\tilde{X}$ st. $\tilde{f}_1(y) = \tilde{f}_1(y) \in \tilde{U}$

and $p|_{\tilde{U}} : \tilde{U} \to U$ is a homeomorphism

since f is continuous $\exists$ open nbhd V of y st. $f(V) \subset U$

now $\tilde{f}_1|_V = (p|_{\tilde{U}})^{-1} \circ f|_V = \tilde{f}_2|_V$

$\therefore V \subset A$ and A open

if $y \notin A$, then with U as above, $\exists \tilde{U}_1, \tilde{U}_2$ open in $\tilde{X}$ st.

$\tilde{f}_i(y) \subset \tilde{U}_i$ and $p|_{\tilde{U}_i} : \tilde{U}_i \to U$ a homeomorphism

clearly $\tilde{U}_1 \cap \tilde{U}_2 = \emptyset$

$\therefore$ if V is as above, then $\tilde{f}_i(V) \subset \tilde{U}_i$

so $X - A$ open

$\therefore$ by connectedness of X, $A = X$ ∎

Two covering spaces $(\tilde{X}_i, p_i)$, $i = 1, 2$, of X are isomorphic if $\exists$ a homeomorphism $h : \tilde{X}_1 \to \tilde{X}_2$ st. $p_2 \circ h = p_1$

$$\begin{tikzcd} \tilde{X}_1 \arrow[rr, "h"] \arrow[dr, "p_1"'] & & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

note: this is an equivalence relation

Cor 25:

Suppose $(\tilde{X}_i, p_i)$, $i = 1, 2$, are path connected, locally path connected covering spaces of X, $x_0 \in X$, $\tilde{x}_i \in p_i^{-1}(x_0)$

a) if $(p_1)_*(\pi_1(\tilde{X}, \tilde{x}_1)) \subset (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$

then p_1 lifts to a covering space $p : \tilde{X}_1 \to \tilde{X}_2$ taking $\tilde{x}_1$ to $\tilde{x}_2$.

b) $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ are isomorphic covering spaces of X by an isomorphism taking $\tilde{x}_1$ to $\tilde{x}_2$

$\iff$

$(p_1)_*(\pi_1(\tilde{X}, \tilde{x}_1)) = (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))$

$$\boxed{\begin{array}{c}\text{c) } (\tilde{X}_1, p_1) \text{ and } (\tilde{X}_2, p_2) \text{ are isomorphic covering spaces of } X \\ \Leftrightarrow \\ (p_1)_*(\pi_1(\tilde{X}_1, \tilde{x}_1)) \text{ is conjugate to } (p_2)_*(\pi_1(\tilde{X}_2, \tilde{x}_2))\end{array}}$$

Proof: a) by Th^m 23 we get a lift $p: \tilde{X}_1 \to \tilde{X}_2$ taking $\tilde{x}_1$ to $\tilde{x}_2$

$$\begin{tikzcd} \tilde{X}_1 \arrow[dr, "p"] \arrow[dd, "p_1"'] & \\ & \tilde{X}_2 \arrow[dl, "p_2"] \\ X & \end{tikzcd}$$

we now show p a covering map

$x \in \tilde{X}_2$ let U be a nbhd of $p_2(x)$ in X that is evenly covered by p_1 and p_2

so $\exists$ a unique $\tilde{U}$ in $\tilde{X}_2$ s.t. $x \in \tilde{U}$ and $p_2|_{\tilde{U}}: \tilde{U} \to U$ is a homeomorphism

let $p^{-1}(\tilde{U}) = \bigcup_\alpha \tilde{U}_\alpha$, clearly $\bigcup_\alpha \tilde{U}_\alpha \subset \tilde{p}_1^{-1}(U)$ so $p_1|_{\tilde{U}_\alpha}: \tilde{U}_\alpha \to U$ a homeo.

so $p|_{U_\alpha} = p_2^{-1}|_{U_\alpha} \circ p_1|_{U_\alpha}$ is a homeomorphism $\tilde{U}_\alpha \to \tilde{U}$

$\therefore$ each point in p has an evenly covered nbhd.

b) ($\Rightarrow$) clear

($\Leftarrow$) let $\tilde{p}_1: \tilde{X}_1 \to \tilde{X}_2$ be lift of p_1

$\tilde{p}_2: \tilde{X}_2 \to \tilde{X}_1$ be lift of p_2

$$\begin{tikzcd} \tilde{X}_2 \arrow[r, "\tilde{p}_2"] \arrow[dr, "p_2"'] & \tilde{X}_1 \arrow[r, "\tilde{p}_1"] \arrow[d, "p_1"] & \tilde{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

note: $\tilde{p}_1 \circ \tilde{p}_2: \tilde{X}_2 \to \tilde{X}_2$ takes $\tilde{x}_2$ to $\tilde{x}_2$ and is a lift of p_2 to $\tilde{X}_2$

$$\begin{tikzcd} & \tilde{X}_2 \arrow[d, "p_2"] \\ \tilde{X}_2 \arrow[ur, "\tilde{p}_1 \circ \tilde{p}_2"] \arrow[r, "p_2"] & X \end{tikzcd}$$

but so is $id_{\tilde{X}_2}: \tilde{X}_2 \to \tilde{X}_2$ $\therefore$ by lemma 24, $\tilde{p}_1 \circ \tilde{p}_2 = id_{\tilde{X}_2}$

similarly $\tilde{p}_2 \circ \tilde{p}_1 = id_{\tilde{X}_1}$

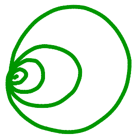
$\therefore$ $\tilde{p}_1$ and $\tilde{p}_2$ are homeomorphisms.

c) clear from lemma 22 and b)

A space X is <u>semilocally simply connected</u> if each point $x \in X$ has a neighborhood U s.t. $\pi_1(U,x) \longrightarrow \pi_1(X,x)$ is trivial

induced by inclusion

<u>example</u>:

$X =$  $= \bigcup_{n=1}^{\infty} \underbrace{S_{\frac{1}{n}}(\tfrac{1}{n},0)}$

circle of radius $\frac{1}{n}$ about $(\frac{1}{n},0)$

is <u>not</u> semi-locally simply connected

but CW-complexes and manifolds <u>are</u>

<u>Th^{m} 26:</u>

let X be a path connected, locally path connected, and semilocally simply connected space $x_0 \in X$, Then there is a one-to-one correspondence

$$\left\{ \begin{array}{l} \text{base point preserving isomorphism} \\ \text{classes of coverings } (\tilde{X},p,\tilde{x}) \text{ of } (X,x_0) \end{array} \right\} \longleftrightarrow \left\{ \text{subgroups of } \pi_1(X,x_0) \right\}$$

$$(\tilde{X},p,\tilde{x}_0) \longmapsto p_*(\pi_1(\tilde{X},\tilde{x}_0)) \cong \pi_1(\tilde{X},\tilde{x}_0)$$

$$(\tilde{X}_H,p_H,\tilde{x}_H) \longleftarrow H$$

such that

1) if $H < K$, then $(\tilde{X}_H,p_H,\tilde{x}_H)$ is also a cover of $(\tilde{X}_K,p_K,\tilde{x}_K)$

2) if p_1 in $(\tilde{X}_1,p_1,\tilde{x}_1)$ lifts to a cover of $(\tilde{X}_2,p_2,\tilde{x}_2)$ taking $\tilde{x}_1$ to $\tilde{x}_2$ then $(p_1)_*(\pi_1(\tilde{X}_1,\tilde{x}_1)) < (p_2)_*(\pi_1(\tilde{X}_2,\tilde{x}_2))$

3) $[\pi_1(X,x_0):H] = n \Leftrightarrow (\tilde{X}_H,p_H,\tilde{x}_H)$ is a cover of degree n

in addition, we have the one-to-one correspondence

$$\left\{ \begin{array}{l} \text{isomorphism classes of} \\ \text{coverings } (\tilde{X},p) \text{ of } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{conjugacy classes of} \\ \text{subgroups of } \pi_1(X,x_0) \end{array} \right\}$$

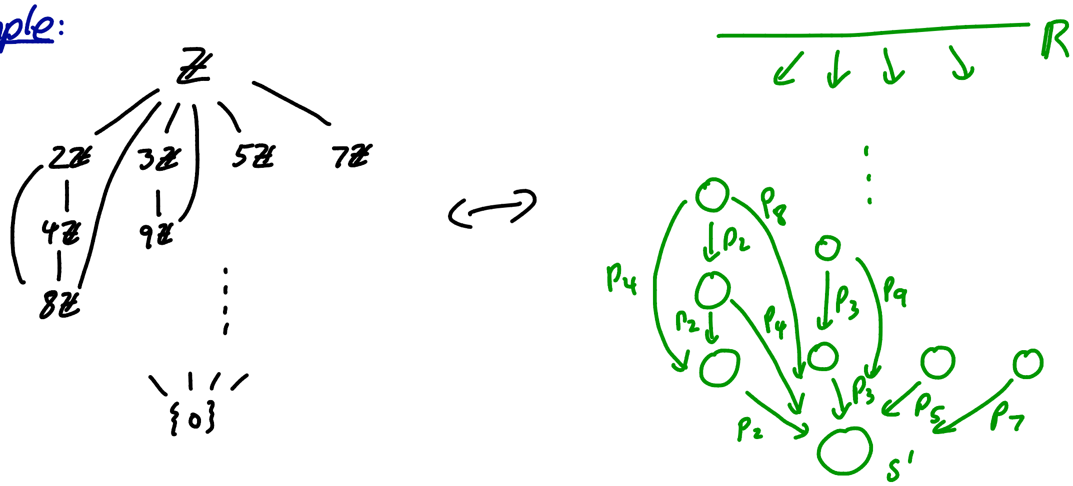
This is an amazing th^{m}! There is a "lattice" of subgroups of $\pi_1(X,x_0)$ and a "lattice" of covering spaces of X

These lattices are the <u>same</u>!

we will see there is more to this correspondance later

The unique simply connected cover of X is called the <u>universal cover</u>

example:



Proof:

note 1) and 2) follow from Cor 25 once we have the one-to-one corresp.

3) follows from lemma 21

also, once we have a well-defined map $H < \pi_1(X,x_0) \mapsto (\tilde{X}_H, p_H, \tilde{x}_H)$

such that $p_*(\pi_1(\tilde{X}_H,\tilde{x}_H)) = H$

the fact that the 1^{st} correspondance is one-to-one follows from Cor 25,b)

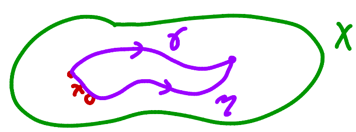
and the 2^{nd} correspondance from Cor 25,c)

so we are left to construct $(\tilde{X}_H, p_H, \tilde{x}_H)$ given $H < \pi_1(X,x_0)$

to this end we call two paths $\gamma, \eta : [0,1] \to X$ based at x_0, <u>H-equivalent</u>

if 1) $\gamma(1) = \eta(1)$ and

2) $[\gamma * \bar{\eta}] \in H$



exercise: 1) This is an equivalence relation

2) if $H = \{e\}$ then this is just homotopy rel end points

let $\langle \gamma \rangle$ denote the equivalence class of γ

Set $\tilde{X}_H = \{\langle\gamma\rangle \mid \gamma \text{ a path in } X \text{ based at } x_0\}$

(this is just a set, but we put a topology on it later)

$p_H : \tilde{X}_H \to X : \langle\gamma\rangle \mapsto \gamma(1)$

$\tilde{x}_H = \langle e_{x_0} \rangle$

note: p_H is onto since any point $x \in X$ is connected to x_0 by a path

We want to define a topology on $\tilde{X}_H$, but first we need to undestand something about the topology on X

Claim: $\mathcal{U} = \{ U \subset X : U \text{ open, path-connected and } \pi_1(U,x) \to \pi_1(X,x) \text{ trivial, some } x \in U \}$ is a basis for the topology on X

(recall, a collection of open sets in X is a basis for the topology on X if $\forall x \in X$ and open set U with $x \in U$ $\exists$ an open set O in the collection s.t. $x \in O \subset U$.
i.e. any open set is a union of sets in the collection)

Pf: note: if $\pi_1(U,x) \to \pi_1(X,x)$ trivial

then $\pi_1(U,y) \to \pi_1(X,y)$ trivial $\forall y \in U$ since

$$\begin{tikzcd} \pi_1(U,x) \arrow[r] \arrow[d, "\phi_h", "\cong"'] \arrow[dr, phantom, "\circ"] & \pi_1(X,x) \arrow[d, "\phi_h"', "\cong"] \\ \pi_1(U,y) \arrow[r] & \pi_1(X,y) \end{tikzcd}$$
h path y to x

also, if $U \in \mathcal{U}$ and $V \subset U$ is open and path connected

then
$$\begin{tikzcd} \pi_1(V,x) \arrow[r] \arrow[rr, bend right, "\text{trivial}"'] & \pi_1(U,x) \arrow[r] & \pi_1(X,x) \end{tikzcd}$$

$\therefore V \in \mathcal{U}$

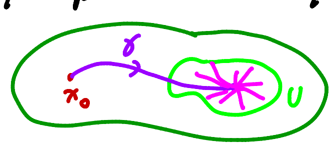
now X semilocally simply connected says for any $x \in X$ and open set U containing x, $\exists$ open set V s.t.
$x \in V$ and $\pi_1(V,x) \to \pi_1(X,x)$ trivial
so $U \cap V$ open set containing x and
$\pi_1(U \cap V, x) \to \pi_1(X,x)$ trivial

X locally path connected $\Rightarrow \exists$ open W s.t. $x \in W \subset U \cap V$ and
W is path connected
from above $\pi_1(W,x) \to \pi_1(X,x)$ trivial
so $W \in \mathcal{U}$ and $\mathcal{U}$ is a basis for topology on X. ✓

for each $U \in \mathcal{U}$ and γ a path x_0 to a point in U

set $U_\gamma = \{ \langle \gamma * \eta \rangle \mid \eta \text{ a path in } U \text{ s.t. } \eta(0) = \gamma(1) \}$

note $U_\gamma \subset \tilde{X}_H$



Claim: $\{U_\gamma\}_{\substack{U\in\mathcal{U} \\ \gamma \text{ path } x_0 \text{ to} \\ \text{pt in } U}}$ forms a basis for a topology on $\tilde{X}_H$

(recall, a collection of sets in a set form a <u>basis for a topology</u> if given any two sets U, V in collection and a point $x \in U \cap V$, $\exists\, W$ in collection s.t. $x \in W \subset U \cap V$ and the set is the union all elts in collection)

Pf:

<u>note</u>: 1) if $\langle\gamma\rangle = \langle\delta\rangle$, then $U_\gamma = U_\delta$ (so can write $U_{\langle\gamma\rangle}$)

indeed, if $\gamma \sim \delta$, then $\gamma * \eta \sim \delta * \eta$ $\forall\, \eta$ path in U with $\eta(0) = \gamma(1)$

since $(\gamma * \eta) * (\overline{\delta * \eta}) = \gamma * \eta * \bar{\eta} * \bar{\delta}$
$\sim \gamma * \bar{\delta}$

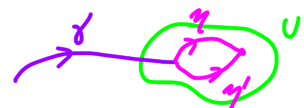
so $[\gamma * \eta * \overline{\delta * \eta}] \in H$.

2) $p : U_{\langle\gamma\rangle} \to U$ is onto (since U path connected)

3) $p : U_{\langle\gamma\rangle} \to U$ is one-to-one

indeed, if $p(\langle\gamma * \eta\rangle) = p(\langle\gamma * \eta'\rangle)$, then $(\gamma * \eta)(1) = (\gamma * \eta')(1)$

so $\eta * \bar{\eta}'$ is a loop in U



based at x $\therefore$ $\eta * \bar{\eta}'$ is homotopically trivial in X

$\Rightarrow \eta \sim \eta'$ rel end points

$\Rightarrow \gamma * \eta \sim \gamma * \eta'$ rel end points

$\Rightarrow [(\gamma * \eta) * (\overline{\gamma * \eta'})] = [e_{x_0}] \in H$

so $\langle\gamma * \eta\rangle = \langle\gamma * \eta'\rangle$ ✓

4) If $\langle\gamma'\rangle \in U_{\langle\gamma\rangle}$, then $U_{\langle\gamma'\rangle} = U_{\langle\gamma\rangle}$

indeed, by hypothesis $\exists\, \eta$ a path in U s.t. $\langle\gamma'\rangle = \langle\gamma * \eta\rangle$

so we can take $\gamma * \eta$ to represent γ' by 1)

if $\langle\delta\rangle \in U_{\langle\gamma'\rangle}$, then $\delta = (\gamma * \eta) * \eta' = \gamma * (\eta * \eta')$

so $\langle\delta\rangle \in U_{\langle\gamma\rangle}$

similarly $\langle\delta\rangle \in U_{\langle\gamma\rangle} \Rightarrow \langle\delta\rangle \in U_{\langle\gamma'\rangle}$ ✓

now if $\langle\delta\rangle \in U_{\langle\gamma\rangle} \cap V_{\langle\gamma'\rangle}$, then $U_{\langle\gamma\rangle} = U_{\langle\delta\rangle}$ and $V_{\langle\gamma'\rangle} = V_{\langle\delta\rangle}$

so if $W \in \mathcal{U}$ s.t. $W \subset U \cap V$ and $\delta(1) \in W$

then $W_{\langle\delta\rangle} \subset U_{\langle\delta\rangle} \cap V_{\langle\delta\rangle} = U_{\langle\gamma\rangle} \cap V_{\langle\gamma'\rangle}$

clearly $\tilde{X}_H$ is a union of all $U_{\langle\gamma\rangle}$'s

so we have a basis for a topology on $\tilde{X}_H$ ✓

Claim: with above topology $(\tilde{X}_H, p_H)$ is a covering space of X

Pf: <u>note:</u> $\forall\, U \in \mathcal{U}$, γ paths x_0 to pt in U

$p_H|_{U_{\langle\gamma\rangle}} : U_{\langle\gamma\rangle} \to U$ a homeomorphism

indeed, p_H is a bijection by 2) and 3)

$p_H|_{U_{\langle\gamma\rangle}}$ is continuous since any basic open set $V \subset U$

and any path δ from x_0 to pt in V

$$\left(p_H|_{U_{\langle\gamma\rangle}}\right)^{-1}(V) = V_{\langle\delta\rangle} \quad \therefore \text{ open}$$

the above argument also shows that basic open sets in $U_{\langle\gamma\rangle}$ map to basic open sets in U

$\therefore\ p_H|_{U_{\langle\gamma\rangle}}$ a homeomorphism ✓

note this implies p is continuous

now if $x \in X$, then let $U \subset \mathcal{U}$ be a set containing x

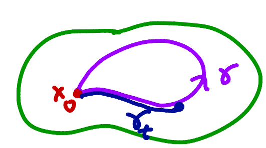
$p^{-1}(U) =$ union of $U_{\langle\gamma\rangle}$ as γ runs through all paths x_0 to pt in U

and $p|_{U_{\langle\gamma\rangle}} : U_{\langle\gamma\rangle} \to U$ homeomorphism. ✓

Claim: $(p_H)_*(\pi_1(\tilde{X}_H, \tilde{x}_0)) = H$

Pf: if $[\gamma] \in H$, then let $\gamma_t(s)$ be the path

$$\gamma_t : [0,1] \to X, \quad s \mapsto ts$$



<u>note:</u> $\tilde{\gamma} : [0,1] \to \tilde{X}_H$, $t \mapsto \langle\gamma_t\rangle$ is a loop since $\tilde{\gamma}(0) = \langle e_{x_0}\rangle$

and $\tilde{\gamma}(1) = \langle\gamma\rangle = \langle e_{x_0}\rangle$

since $[\gamma] \in H$

moreover $p_H \circ \tilde{\gamma} = \gamma$

$\therefore [\gamma] = \text{image}(p_H)_*$

now if $[\gamma] \notin H$, then the path

$$\tilde{\gamma} : [0,1] \longrightarrow \tilde{X}_H$$
$$t \longmapsto \langle \gamma_t \rangle$$

is clearly the lift of γ based at $\tilde{x}_H$

and $\tilde{\gamma}(1) = \langle \gamma \rangle$

but $\langle \gamma \rangle \neq \langle e_{x_0} \rangle = \tilde{x}_H$ since $[\gamma] \notin H$

$\therefore [\gamma] \notin \text{image}\,(p_H)_*$ by lemma 21, 2) ▦

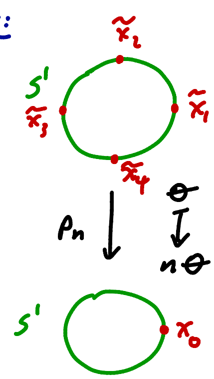
let $p : \tilde{X} \to X$ be a covering space

a <u>deck transformation</u> or <u>covering transformation</u> is a covering space isomorphism $f : \tilde{X} \to \tilde{X}$

the set $G(\tilde{X})$ of deck transformations clearly is a group under the operation of composition

<u>examples:</u>

1)



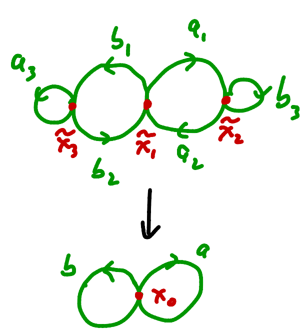
$$\phi_k : S^1 \to S^1 \qquad \text{is a covering transformation}$$
$$\theta \mapsto \frac{2\pi k}{n}$$

if f is any deck transformation then $f(\tilde{x}_1) = \tilde{x}_i$ for some i, but $\phi_i(\tilde{x}_1) = \tilde{x}_i$ too

so by lemma 24, $f = \phi_i$ (since covering transforms are lifts of p_n)

so $G(\tilde{X}) = \mathbb{Z}/n\mathbb{Z}$

2)



if we lift b to $\tilde{x}_3$ then it is a loop but if we lift to $\tilde{x}_2$ or $\tilde{x}_1$ it is a path

so no deck trans. taking $\tilde{x}_3$ to $\tilde{x}_1$ or $\tilde{x}_2$

similarly can't send $\tilde{x}_2$ to $\tilde{x}_1$ or $\tilde{x}_3$

so any dech trans fixes all $\tilde{x}_i$ $\therefore$ is identity

$\therefore$ $G(\tilde{X}) = \{1\}$

A covering space $p: \tilde{X} \to X$ is called normal if $\forall x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$

$\exists \phi \in G(\tilde{X})$ s.t. $\phi(\tilde{x}) = \tilde{x}'$

so example 1) is normal but example 2) is not.

Th^{m} 27:

let $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a path connected, locally path connected covering space of a space X

let $H = p_*(\pi_1(\tilde{X}, \tilde{x}_0)) < \pi_1(X, x_0)$, then

1) $(\tilde{X}, p)$ is normal $\Leftrightarrow$ H is a normal subgroup of $\pi_1(X, x_0)$

2) $G(\tilde{X}) \cong N(H)/H$ where $N(H)$ is the "normalizer" of H, i.e. largest subgroup of $\pi_1(X, x_0)$ containing H as a normal subgroup

Remark: If $p: \tilde{X} \to X$ normal, then $G(\tilde{X}) = \pi_1(X, x_0) / p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

in particular, for the universal cover $p: \tilde{X} \to X$

$G(\tilde{X}) = \pi_1(X, x_0)$

Proof:

1) $(\Rightarrow)$ let $[\gamma] \in \pi_1(X, x_0)$

and $\tilde{\gamma}$ a lift of γ based at $\tilde{x}_0$

set $\tilde{x}_1 = \tilde{\gamma}(1)$

(proof of)

from lemma 22 $[\gamma]\, p_*(\pi_1(\tilde{X}, \tilde{x}_1))\, [\gamma]^{-1} = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

by hypothesis $\exists \phi \in G(\tilde{X})$ s.t. $\phi(\tilde{x}_1) = \tilde{x}_0$

so $\phi_*: \pi_1(\tilde{X}, \tilde{x}_1) \to \pi_1(\tilde{X}, \tilde{x}_0)$ an isomorphism

$\therefore$ $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = p_* \circ \phi_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

and $[\gamma] H [\gamma]^{-1} = H$ ✓

$(\Leftarrow)$ let $\tilde{x}_0$ and $\tilde{x}_1$ be two points in $p^{-1}(x_0)$

and $H_1 = p_*(\pi_1(\tilde{X}, \tilde{x}_1))$

let h be a path in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x}_1$ and $\gamma = p \circ h$

by lemma 22, $[\gamma] H_1 [\gamma]^{-1} = H$

$\therefore H = H_1$ since H is normal

i.e. $p_*(\pi_1(\tilde{X}, \tilde{x}_1)) = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

$\therefore$ by Th^{m} 23 $\exists$ lifts of p to ϕ_1 and ϕ_2

$$\begin{tikzcd}
(\tilde{X}, \tilde{x}_1) \arrow[r, "\phi_1"] \arrow[dr, "p"'] & (\tilde{X}, \tilde{x}_0) \arrow[r, "\phi_2"] \arrow[d, "p"] & (\tilde{X}, \tilde{x}_1) \arrow[dl, "p"] \\
 & (X, x_0) &
\end{tikzcd}$$

<u>note</u>: $\phi_2 \circ \phi_1$ is a lift of p that fixes $\tilde{x}_1$

so is $id_{\tilde{X}}$ $\therefore \phi_2 \circ \phi_1 = id_{\tilde{X}}$

similarly $\phi_1 \circ \phi_2 = id_{\tilde{X}}$

$\therefore \phi_1$ a deck transform taking $\tilde{x}_1$ to $\tilde{x}_0$

<u>exercise</u>: for any $x \in X$ and $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ show $\exists \phi \in G(\tilde{X})$ s.t. $\phi(\tilde{x}) = \tilde{x}'$

2) from above if $[\gamma] H [\gamma]^{-1} = H$ then $\exists \phi \in G(X)$ s.t. $\phi(\tilde{x}_0) = \tilde{\gamma}(1)$ ($\tilde{\gamma}$ lift based at $\tilde{x}_0$)

so we get a map

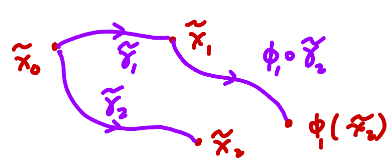
$$\Phi : N(H) \to G(\tilde{X})$$

<u>Claim:</u> Φ a homomorphism

suppose $\phi_i = \Phi([\gamma_i])$ for $[\gamma_i] \in N(H)$ $i = 1, 2$

$\phi_i(\tilde{x}_0) = \tilde{x}_i$ so $\tilde{\gamma}_i$ path $\tilde{x}_0$ to $\tilde{x}_i$

<u>note</u>: $\tilde{\gamma}_1 * (\phi_1 \circ \tilde{\gamma}_2)$ is a path $\tilde{x}_0$ to $\phi_1(\tilde{x}_2)$



and $[p \circ (\tilde{\gamma}_1 * (\phi_1 \circ \tilde{\gamma}_2))] = [\gamma_1 * \gamma_2] = [\gamma_1] \cdot [\gamma_2]$

so $\phi_1 \circ \phi_2$ is $\Phi([\gamma_1] \cdot [\gamma_2])$

<u>Claim:</u> Φ is surjective

let $\phi \in G(\tilde{X})$ take a path h in $\tilde{X}$ from $\tilde{x}_0$ to $\tilde{x} = \phi(\tilde{x}_0)$

$\gamma = p \circ h$ is a loop in X

and from above $[\gamma] H [\gamma]^{-1} = H$ so $[\gamma] \in N(H)$

and $\Phi([\gamma]) = \phi$

Claim: $\ker \Phi = H$

if $[\gamma] \in H$, then $\tilde{\gamma}(1) = \tilde{x}_0$ so $\Phi([\gamma]) = id_{\tilde{X}}$

$\therefore H \subset \ker \Phi$

if $[\gamma] \in \ker \Phi$, then $\tilde{\gamma}(1) = \tilde{x}_0$ and so $\tilde{\gamma}$ a loop

$\therefore [\gamma] \in H$

a group action on a topological space X is a pair (G, ρ) where

G is a group, and

$\rho: G \to Homeo(X)$ is a homomorphism

group of homeomorphisms

if G acts on X then we can form the quotient space X/G

where two points x_1, x_2 are identified if $\exists g \in G$ s.t. $\rho(g)(x_1) = x_2$

this is called the orbit space

Th^{m} 28:

let G be a group action on X such that

* $\forall x \in X, \exists$ a nbhd U of x so that $g_1 U \cap g_2 U \neq \emptyset \Rightarrow g_1 = g_2$

then 1) $p: X \to X/G$ is a normal covering space

2) $G \cong G(X \to X/G)$ if X is path connected

3) $G \cong \pi_1(X/G) / p_*(\pi_1(X))$ if X is path connected and locally path connected.

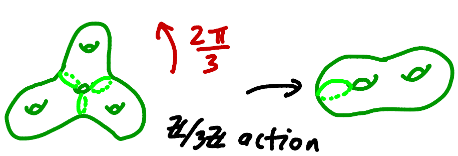
Proof: fairly easy

exercise or see Hatcher

exercise: if G is finite and G acts freely on X (i.e. has no fixed points)

then the action on X satisfies *

examples:

1) 

2) $S^n \to \mathbb{RP}^n$

$\mathbb{Z}/2\mathbb{Z}$ action